Problem 3

If $u_1(x, y)$ and $u_2(x, y)$ satisfy equation (1.1), then is it true that the sum satisfies it?; if yes, prove it.

[This is bad punctuation. Take out the semicolon and capitalize the first letter of "if."]

Solution

Equation (1.1) in the textbook is the general form of a second-order linear PDE with two independent variables.

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{1.1}$$

Suppose that $u_1(x, y)$ and $u_2(x, y)$ are solutions to this equation.

$$A\frac{\partial^2 u_1}{\partial x^2} + B\frac{\partial^2 u_1}{\partial x \partial y} + C\frac{\partial^2 u_1}{\partial y^2} + D\frac{\partial u_1}{\partial x} + E\frac{\partial u_1}{\partial y} + Fu_1 = G$$

$$A\frac{\partial^2 u_2}{\partial x^2} + B\frac{\partial^2 u_2}{\partial x \partial y} + C\frac{\partial^2 u_2}{\partial y^2} + D\frac{\partial u_2}{\partial x} + E\frac{\partial u_2}{\partial y} + Fu_2 = G$$

Add the respective sides of this equation.

$$A\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_2}{\partial x^2}\right) + B\left(\frac{\partial^2 u_1}{\partial x \partial y} + \frac{\partial^2 u_2}{\partial x \partial y}\right) + C\left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_2}{\partial y^2}\right) + D\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x}\right) + E\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial y}\right) + F(u_1 + u_2) = 2G$$

Use the fact that the derivative is a linear operator.

$$A\frac{\partial^2}{\partial x^2}(u_1+u_2) + B\frac{\partial^2}{\partial x \partial y}(u_1+u_2) + C\frac{\partial^2}{\partial y^2}(u_1+u_2) + D\frac{\partial}{\partial x}(u_1+u_2) + E\frac{\partial}{\partial y}(u_1+u_2) + F(u_1+u_2) = 2G$$

Because the right side is $2G$ and not G, the sum of $u_1(x, y)$ and $u_2(x, y)$ does not satisfy equation (1.1).